Chapter I.

1. Definition.

$\boxed{\text{Def 1}}$. The n-th Weyl algebra A_n is the K-subalgebra of $\mathrm{End}_K(K[x])$ generated by operators.

$\hat{x}_1, \dots \hat{x}_n, \partial_1, \dots \partial_n$ where.

$\hat{x}$ is defined as: $\forall f \in K[x]$ $\quad \hat{x}_i(f) = x_i \cdot f$

∂_i is defined as: $\forall f \in K[x]$. $\quad \partial_i(f) = \dfrac{\partial f}{\partial x_i}$

Define $A_0 = K$ $\quad$ E.g. $n \geq m \Rightarrow$ well-defined. $\quad A_1 \curvearrowright K[x_1]$ ✓ $\quad A_2 \curvearrowright K[x_1]$ $(\partial_2, \hat{x}_2, ?)$

$\boxed{\text{Rmk2}}$ A_n is not commutative

A_m is subalg. of A_n naturally.

E.g. $[\partial_i, x_i] = 1$, i.e. $\quad \partial_i(x_i f) = x_i \cdot \dfrac{\partial f}{\partial x_i} + f$

not the number "1", it means identity operator.

E.g. $[\partial_i, x_j] = \delta_{ij} \cdot 1$

$[\partial_i, \partial_j] = [x_i, x_j] = 0$, $\quad 1 \leq i, j \leq n$

2. Canonical Form.

$\boxed{\text{Def 3}}$. Canonical basis

Def 3.1. multi-index if an element of $\mathbb{N}^n$, say

$\alpha = (\alpha_1, \dots \alpha_n)$, $\quad \alpha_1, \dots \alpha_n \in \mathbb{N} = \{0, 1, \dots\}$

x^α means the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$

Def 3.2. degree. deg of the monomial is the length

$|\alpha|$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$

Ref 3.3. Factorial of a multi-index $\beta \in \mathbb{N}^n$ is: $\beta! = \beta_1! \cdots \beta_n!$ proof:

Ref 3.4. The set $B = \{ x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n \}$ is a basis of A_n called canonical basis

We need to prove this. but later!

Ref 3.5. If an element of A_n is writen as linear combination of this basis (Ref 3.3), we say that it is in canonical form.

$$\beta! = \partial^\beta x^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} x_1^{\beta_1} \cdots x_n^{\beta_n}$$

$$= \beta_1! \, \partial_2^{\beta_2} \cdots \partial_n^{\beta_n} x_2^{\beta_2} \cdots x_n^{\beta_n} = \cdots = \beta_1! \cdots \beta_n!$$

Lemma: Let $\alpha, \beta \in \mathbb{N}^n$, assume $|\alpha| \le |\beta|$. Then $\partial^\beta (x^\sigma) = \beta!$ if $\sigma = \beta$, 0 otherwise.

Proof of Ref 3.5: $B = \{ x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n \}$ is a basis of A_n

1° Obviously: $\mathrm{Span}_k\{B\}$ vector space. (as a vector space over k)

2° aim: bring all powers of x's to the left side.

$$(\partial_i \cdot f - f \cdot \partial_i)(g)$$

$$= \partial_i (f \cdot g) - f(\partial_i g)$$

$$= \frac{\partial f \cdot g}{\partial x_i} - f \cdot \frac{\partial g}{\partial x_i}$$

$$= g \cdot \frac{\partial f}{\partial x_i} + f \frac{\partial g}{\partial x_i} - f \frac{\partial g}{\partial x_i} = g \cdot \frac{\partial f}{\partial x_i} = \boxed{\frac{\partial f}{\partial x_i}} \, g$$

polynomial.

or operator generated by some $\hat{x_i}$

3°. Uniqueness: ($\forall D \in A_n$, canonical form is unique

$\Updownarrow$ linear independence

Consider $D = \sum_{\text{fin}} C_{\alpha\beta} x^\alpha \partial^\beta$

if some $C_{\alpha\beta}$ is non-zero then $D \neq 0$

Since. D is a operator. $D \neq 0$ iff $\exists f \in K[x]$ s.t $D(f) \neq 0$

aim: find (at least one) suitable f.

Let σ be a multi-index s.t $C_{\alpha\sigma} \neq 0$ for some index α.

but $C_{\alpha\beta} = 0$, for all indices β s.t. $|\beta| < |\sigma|$

E.g: $D = x_1 x_2 \partial_1 + 2x_1^2 x_2 \partial_1^2 + x_1 x_2^2 \partial_1 \partial_2 + 4 x_1 x_2^2 \partial_1^3 \partial_2^2$

$1 \rightarrow C_{(1,1)(1,0)} \; x^{(1,1)} \partial^{(1,0)}$

$2 \rightarrow C_{(2,1)(2,0)} \; x^{(2,1)} \partial^{(2,0)}$

$1 \rightarrow C_{(1,2)(1,1)} \; x^{(1,2)} \partial^{(1,1)}$

$4 \rightarrow C_{(1,2)(3,2)} \; x^{(1,2)} \partial^{(3,2)}$

So. $\sigma = (1,0)$

So show

$D(x_1) = x_1 x_2$

It just means find "smallest" σ, with condition $\exists \, C_{\alpha\sigma} \neq 0$ so it's a meaningful work. or it makes no sense.

Consider $D(x^\sigma) = \sigma! \sum_\alpha C_{\alpha\sigma} x^\alpha$

sum by α since for $\begin{cases} |\beta| < |\sigma|, & C = 0 \\ |\beta| = |\sigma|, & \text{sum by } \alpha \\ |\beta| > |\sigma|, & 0 \end{cases}$

Now $D(x^\sigma)$ is non-zero for sure. Thus $f = x^\sigma$ is what we want.

$\Rightarrow$ Uniqueness proved.

3. Generators and Relations.

$\boxed{\text{Def 4}}$ free algebra $k\{z_1, \dots, z_{2n}\}$ in $2n$ generators is the set of all finite linear combination of words in $z_1 \dots z_{2n}$

Multiplication of two monomials is simple juxtaposition.

$\boxed{\text{Def 5}}$ Let $J = \langle \{[z_{i+n}, z_i]-1\}_{i=1,\dots,n},\ \{[z_i, z_j]\}_{\substack{j \neq i+n \\ 1 \le i,j \le 2n}} \rangle$ be two-sided ideal.

$J = \{ | \cdot k[Z] + k[Z] \{ \} + \{ | k[Z] \{] + Z \{ \}$

Define: $\phi : k\{z_1, \dots z_{2n}\} \to A_n$

$$z_i \mapsto x_i$$
$$z_{i+n} \mapsto \partial_i \qquad i = 1, 2 \dots n.$$

(surjective. obviously).

$\Rightarrow J \subseteq \ker\phi$

$\boxed{\text{Thm 6}}$. $\hat{\phi} : k\{z_1 \cdots z_{2n}\}/J \longrightarrow A_n$

k-alg. homo.

$$z_i \longmapsto x_i$$
$$z_{i+n} \longmapsto \partial_i \ , \quad i = 1, \dots, n \quad \text{is an iso.}$$

Review the canonical basis of A_n. the image of $\hat{\phi}$ form a basis of A_n as a vector space over k and the monomials are linearly inde. in $k\{z_1 \dots z_{2n}\}/J$

$$\text{monomials in } k\{Z\}/J \ \underset{\text{pre image}}{\overset{\hat{\phi} \text{ surjective}}{\rightleftarrows}} \ \text{canonical basis of } A_n$$

linear ind.

hence $\hat{\phi}$ is an iso. of k-vector spaces.

It's also iso of ring since.

$$\hat{\phi}\left((z^{(\alpha_1,\beta_1)}+J)(z^{(\alpha_2,\beta_2)}+J)\right)$$
$$=\hat{\phi}(z^{(\alpha_1,\beta_1)}\cdot z^{(\alpha_2,\beta_2)}+J)$$
$$= x^{\alpha_1}\partial^{\beta_1}\cdot x^{\alpha_2}\partial^{\beta_2}$$
$$=\hat{\phi}(z^{(\alpha_1,\beta_1)}+J)\cdot\hat{\phi}(z^{(\alpha_2,\beta_2)}+J).$$

$\boxed{\text{Cor 7}}$. Let $m<n$ be positive integers. Chose $f_i\in k[x]$ $1\le i\le n$ as:

for $i\le m$, f_i is a polynomial in the variable $x_{m+1},\dots,x_n$.

$m<i\le n$ $f_i=0$

The map $\sigma: A_n - A_n$ s.t.

$$x_i\mapsto x_i+f_i$$
$$\partial_i\mapsto\partial_i-\sum_{1}^{m}\frac{\partial f_k}{\partial x_i}\partial_k$$
is an auto. of A_n.

E.g.: $n=2$. $m=1$

$f_1=k_2x_2$ $f_2=0$

$\sigma: A_2\to A_2$

$x_1\mapsto x_1+f_1=x_1+k_2x_2$

$x_2\mapsto x_2+f_2=x_2$.

$\partial_1\mapsto\partial_1-\left(\frac{\partial f_1}{\partial x_1}\partial_1+\frac{\partial f_2}{\partial x_1}\partial_2\right)$

$=\partial_1$

$\partial_2\mapsto\partial_2-\left(\frac{\partial f_1}{\partial x_2}\partial_1+\frac{\partial f_2}{\partial x_1}\partial_2\right)$

$=\partial_2-k_2\partial_1$

in a word.

$$\begin{cases} x_1\mapsto x_1+k_2x_2\\ x_2\mapsto x_2\\ \partial_1\mapsto\partial_1\\ \partial_2\mapsto\partial_2-k_2\partial_1\end{cases}$$

proof: (This proof only show that σ is an endo. further details is discussed in chap 4.)

$\to$ check of homo is easy.

Define a homo ψ : $k\{z_1 \cdots z_{2n}\} \longrightarrow A_n$.

$$z_i \longmapsto x_i + f_i$$

$$z_{i+n} \longmapsto \partial_i - \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} \partial_k \quad , \ 1 \le i \le n.$$

choose i, j s.t. $1 \le i, j \le n$ $\quad \psi([z_i, z_j]) = 0$

Calculate $\psi([z_{i+n}, z_j])$

$$= [\psi(z_{i+n}), \psi(z_j)]$$

$$= [\partial_i, x_j + f_j] - \left[\sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} \partial_k, x_j + f_j\right].$$

$$= \delta_{ij} + \frac{\partial f_j}{\partial x_i} - \left(\frac{\partial f_j}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} \frac{\partial f_j}{\partial x_k} \right)$$

$k \le m$. 0 $\qquad$ $k \ge m$ 0.

$$= \delta_{ij}$$

and $\psi([z_{i+n}, z_{j+n}]) = 0$ thus,

$$\begin{tikzcd} & k\{x_1, \dots x_{2n}\} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ A_n \arrow[rr, "\sigma"'] & & A_n. \end{tikzcd}$$

Consider $\tau(x_i) = x_i - f_i$

$$\tau(\partial_i) = \partial_i + \sum_{k=1}^{n} \frac{\partial f_k}{\partial x_i} \partial_k$$

it is the inverse of σ not need to be checked in Ch4. §4.

Chapter 2. Ideal structure.

1. The degree of an operator.

$\boxed{\text{Def 1}}$. $\forall D \in A_n$, The degree of D is the largest length of the multi-indices $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ for which $x^\alpha \partial^\beta$ appears with non-zero coefficient in the canonical form of D, denoted by $\deg(D)$.

$\deg(0) = -\infty$

$\boxed{\text{Cor 2}}$. $\deg(D+D') \leq \max\{\deg(D), \deg(D')\}$.

$\boxed{\text{Cor 3}}$ $\deg(DD') = \deg D + \deg D'$

$\boxed{\text{Cor 4}}$ $\deg[D, D'] \leq \deg(D) + \deg(D') - 2$.

pf: If either D or D' is zero. then it's obvious.

Sps $\deg D, \deg D' \geq 1$ and the formula holds whenever $\deg D + \deg D' < k$

Let $D, D' \in A_n$ s.t $\deg D + \deg D' = k$

By 1. it's sufficient to prove (2) and (3). when D, D' are monomials.

Sps. $D = \partial^\beta$, $D' = x^\alpha$ with $|\alpha| + |\beta| = k$. if $\beta_i \neq 0$

then. $[\partial^\beta, x^\alpha] = \partial_i [\partial^{\beta - e_i}, x^\alpha] + [\partial_i, x^\alpha]\partial^{\beta - e_i}$

By induction. $\deg[\partial^{\beta - e_i}, x^\alpha] \leq |\alpha| + |\beta| - 3$

$\deg[\partial_i, x^\alpha] \leq |\alpha| + 1 - 2 = |\alpha| - 1$.

$\Rightarrow$ by hypothesis. $\deg \partial_i [\partial^{\beta - e_i}, x^\alpha]$, $\deg([\partial_i, x^\alpha]\partial^{\beta - e_i})$

$\leq |\alpha| + |\beta| - 2$ (hypothesis (2).)

Therefore. $\deg[\partial^\beta, x^\alpha] \leq |\alpha| + |\beta| - 2$.

furthermore $\deg(\partial^\beta x^\alpha) = \deg(\underbrace{x^\alpha\partial^\beta}_{|\alpha|+|\beta|} + \underbrace{[\partial^\beta, x^\alpha]}_{\le |\alpha|+|\beta|-2})$

$= \deg x^\alpha \partial^\beta = |\alpha|+|\beta|$

Now let $D = x^\sigma \partial^\beta$ $\quad D' = x^\alpha \partial^\eta$

if $|\alpha| = |\beta| = 0$, ✓

Suppose not this case.

$\partial^\beta x^\alpha = x^\alpha \partial^\beta + P \quad P = [\partial^\beta, x^\alpha] \quad \deg P \le |\alpha|+|\beta|-2$.

$$DD' = x^\sigma \underline{\partial^\beta x^\alpha} \partial^\eta$$
$$= x^\sigma (x^\alpha \partial^\beta + P)\partial^\eta$$
$$= x^{\sigma+\alpha}\partial^{\beta+\eta} + x^\sigma P \partial^\eta$$

By induction $\deg x^\sigma P \partial^\eta \le |\sigma| + |\eta| + (|\alpha|+|\beta|-2) = \deg D + \deg D' - 2$

$\Rightarrow \deg DD' = \deg x^{\sigma+\alpha}\partial^{\beta+\eta} = \deg D + \deg D'$

Similarly. $\quad DD' = x^{\sigma+\alpha}\partial^{\beta+\eta} + Q_1 \quad \deg Q_1 \le \deg D + \deg D' - 2$.

$D'D = x^{\sigma+\alpha}\partial^{\beta+\eta} + Q_2 \quad \deg Q_2 \le \deg D + \deg D' - 2$.

$\Rightarrow [D, D'] = Q_1 - Q_2$

$\Rightarrow \deg [D, D'] \le \deg D + \deg D' - 2$. $\square$

$\boxed{\text{Cor } 5}$

Corollary: A_n is a domain.

Review. domain is a ring with no nonzero devisors.

i.e. if $ab = 0$ then $a = 0$ or $b = 0$.

if $ab=0$, $\deg ab = \deg 0 = -\infty$

$\deg ab = \deg a + \deg b$ it shows either a or $b = 0$.

2. Ideal Structure.

$\boxed{\text{Thm 6}}$ A_n is simple.

Let I be a non-zero two-sided ideal of A_n. Choose an element $D \neq 0$ of smallest degree in I. If $\deg D = 0$ $I = A_n$.

Now sps $\deg D = k > 0$.

Sps (α, β) is a multi-index of length k.

Choose a summand $x^\alpha \partial^\beta$ of D with non-zero coefficient and $\beta_i \neq 0$.

then $[x_i, D] = x^{\alpha+e_i}\partial^\beta - x^{\alpha+e_i}\partial^\beta - x^\alpha \partial^{\beta-e_i} + \cdots$

$\Rightarrow \deg [x_i, D] = k-1 \Rightarrow [x_i, D] \in I$. ↯ minimality.

$\Rightarrow \beta = (0, \cdots 0)$ since $k > 0$.

we must have $\alpha_i \neq 0$, for some $i = 1, 2, \dots n$.

$\Rightarrow [\partial_i, D] = \partial_i x^\alpha - \partial_i x^\alpha - x^{\alpha - e_i} + \cdots$

$\deg [\partial_i, D] = k-1$ ↯.

$\boxed{\text{Cor 7}}$ A_n is not a division ring. In fact only elements in A_n have inverses are the constant.

$DD' = 1 \Rightarrow \deg D + \deg D' = \deg 0 \Rightarrow \deg D = \deg D' = 0$.

$\boxed{\text{Cor 8}}$. A_n is not a left principal ideal ring.

E.g. (∂_1, ∂_2) is not principal. Exe. 4.1

[Cor 9] every left ideals generated by 2 elements
hard to prove.

3. Positive characteristic.

E.g. Consider $R_1 = \mathbb{Z}_p[\partial]$

calculate $\partial^p(x^k)$.

If $k < p$. $\partial^p(x^k) = 0$.

if $k \geq p$. $\partial^p(x^k) = \underbrace{k(k-1)\cdots(k-p+1)}_{p!}\, x^{k-p} = 0$

$\Rightarrow \partial^p = 0$. $\Rightarrow$ nilpotent element $\Rightarrow$ not a domain.

E.g. consider $R_2 = \mathbb{Z}_p[z_1, z_2]$. $\Rightarrow [z_2, z_1] = 1$

This is a domain but. it's not a simple ring.

$f \in \mathbb{Z}_p[z_1]$ then $[z_2, f] = \frac{\partial f}{\partial z_1}$

in particular. $[z_2, z_1^p] = p z_1^{p-1} = 0$.

$\Rightarrow z_1^p$ commutes with every elements of R_2.

$\Rightarrow (z_1^p)$ is a two sided ideal. $\Rightarrow R_2$ is not a simple ring.

Chap3. Rings of differential operators.

1. Definitions:

$\boxed{\text{Def 1}}$. order of an operator. define it inductively:

$P \in End_k(R)$ has order zero if $[a, P] = 0$, $\forall a \in R$.

Suppose we have defined operators of order $< n$.

Another operator $P \in End_k R$ has order n if:

it dose not have order less than n and $[a, P]$ has order less than n $\forall a \in R$.

$\boxed{\text{Cor 2}}$ $D^n(R)$ is a k-vector space.

Just to check 8 rules of vector space.

$\boxed{\text{Def 3}}$. Derivation of k-algebra R is a linear operator D of R which satisfies Leibniz's rule:

$$D(ab) = aD(b) + bD(a) \quad \forall a, b \in R.$$

the k-vector space of all derivations of R denoted by $Der_k(R)$.

$Der_k(R) \subseteq End_k(R)$.

And define. if $D \in Der_k(R)$ and $a \in R$,

$$aD : (aD)(b) = a\,D(b) \quad \forall b \in R.$$

Then the vector space $Der_k(R)$ is a left R-module under this action.

$\boxed{\text{Lem 4}}$

The operators of order ≤ 1 correspond to the elements of $\mathrm{Der}_k(R) + R$.
The elements of order 0 are the elements of R.

Proof: Show the 2nd one first.

Consider. if $\forall a \in R$ $[a, p] = 0$

it shows $\forall a \in R$. $ap = pa$

$\Rightarrow \forall b \in R,$ $\quad a p(b) = p(ab)$

and then direct ...

since $p \in \mathrm{End}_k R$. $p(ab) = p(a) p(b)$

$\Rightarrow \quad (a - p(a)) p(b) = 0$

$\Rightarrow p = 0$ or $p(a) = a \Rightarrow p \in \mathrm{End}_R R = R$.

if $p \in R$
$[a, p] = 0$
$\forall a \in R$ is trivial.

$\Rightarrow$. $D^0_K(R) = \mathrm{End}_R R = R$.

Now consider $Q \in D^1(R)$ and Let

$p = Q - Q(1)$.

$p(1) = 0 \Rightarrow$ order p has order ≤ 1.

(If ord $p \geq 2$. then $\exists a = 1$ s.t. $\forall b \in R$ $[b, [a, p]] = 0$ ↯.)

Hence $[p, a]$ has ord 0, $\forall a \in R$.

$\Rightarrow \forall b \in R$. $[[p, a], b] = 0$, i.e.

$[p, a] b - b [p, a]$
$= pab - apb - bpa + bap = 0$

applying to 1. $\quad p(ab) - a p(b) - b p(a) = 0$ $\quad (p(1) = 0)$

$\Rightarrow$ P is a Derivation of R.

$Q = P + Q(1) \in \mathrm{Der}_k(R) + R$. ($Q \in D^1(R)$

$Q(1) \in D^0(R) = R$.)

Done.

$\boxed{\text{Def 5}}$. Ring of differential operators. $D(R)$ of k-alg. R is. $D(R) = \bigcup D^i(R)$, $\subseteq \mathrm{End}_k(R)$.

all operators of $\mathrm{End}_k(R)$ of finite order. with the operations of sum and composition of operators. trivial.

To make it well-defined, need to check sum multiplication of finite order has finite order.

$\boxed{\text{prop 6}}$. Let $P \in D^n(R)$ and $Q \in D^m(R)$, then $P \cdot Q \in D^{m+n}(R)$

induction on $m+n$.

① $m+n=0$. ✓

② suppose $m+n<k$ ✓

③ If $m+n=k$. $\forall a \in R$.

$$[PQ, a] = PQa - aPQ = PQa - PaQ + PaQ - aPQ$$
$$= P[Q,a] + [P,a]Q$$

$P, Q \in D^m(R)$. $D^n(R)$

$[Q, a] \in D^{n-1}(R)$ $[P, a] \in D^{m-1}(R)$

By inductive hypothesis $P[Q,a]$, $[P,a]Q \in D^{m+n-1}(R)$

$\Rightarrow [PQ, a] \in D^{m+n-1}(R)$

$\Rightarrow PQ \in D^{m+n}(R)$. $\square$.

$\boxed{\text{Prop 7}}$. Every derivation of $k[X] = k[x_1 \cdots x_n]$ is of the form $\sum_i^n f_i \partial_i$, for some $f_1 \dots f_n \in k[X]$.

proof:

Let $D \in Der_k(k[X])$

Then. $D(x_i^k) = k x_i^{k-1} D(x_i)$, for $i = 1, \dots, n$. Hence.

$(D - \sum_{i=1}^n D(x_i)\partial_i)(x_1^{s_1} \cdots x_n^{s_n}) = 0$.

$\Rightarrow D \equiv \sum_{i=1}^n D(x_i)\partial_i \Rightarrow D = \sum_{i=1}^n f_i \partial_i$, $f_i = D(x_i)$.

I didn't understand this. I think this need use Kahler differentials.

2. The Weyl algebra.

$\boxed{\text{Lem 8}}$. Let $P \in D(k[X])$. If $[P, x_i] = 0$, $\forall i = 1, \dots n$. then $P \in k[X]$.

(proof. Lem 1.1)

$\Downarrow$

Let $f = x^\alpha$, assume $\alpha_i \neq 0$.

$[P, x^\alpha] = \underbrace{[P, x_i]x^{\alpha - e_i} + x_i [P, x^{\alpha - e_i}]}_{0}$ by induction $\Rightarrow [P, x^\alpha] = 0$.

$\Downarrow$

$\forall f \in k[X]$, $\forall P \in k[X] \subseteq D(k[X])$, $[P, f] = 0$.

$\boxed{\text{Lem 9}}$. Define. $C_r = \{\sum_\alpha f_\alpha \partial^\alpha, |\alpha| \leq r\} \subseteq A_n$.

$C_r = C_{r+1} \cap D^r(k[X])$.

$C_1 = \mathrm{Der}_k(k[X]) + k[X]$ $\quad C_0 = k[X]$.

Let $P_1 \cdots P_n \in C_{r-1}$ and assume $[P_i, x_j] = [P_j, x_i]$

$1 \le i, j \le n$. Then $\exists\, Q \in C_r$ s.t. $P_i = [Q, x_i]$, $\forall i = 1 \ldots n$.

$\boxed{\text{Thm 10}}$ The ring of differential operators of $k[X]$ is $A_n(k)$.

Beside this $D^k(k[X]) = C_k$.

$\boxed{\text{Big Picture}}$

At the first 3 chapters, our final goal is to show

we will show

$$D(k[X]) = A_n(k) \quad , k \text{ is a field (WHY)}$$

That is, the "differential ring" over $k[X]$

is just the "Weyl algebra" over $k[X]$

Naturally we will ask 3 QUESTIONS

1°. where is Weyl algebra ?

2°. Where is Differential ring over a ring ($k[X]$) ?

3°. Why they are exactly "the same" ?

In 1st chapter. we answer the first question.

And as it is a "algebra" we need to know something

about its structure. Most importantly. the ideal structure

This it the 2ed chapter. In the 3rd chapter we deal

with our final goal.

The contents is following:

Chap 1 Introduction of Weyl algebra.

definitions.

Canonical form. $\hookrightarrow$ how to describe it in a standard way.

Generators. relations $\Rightarrow$ Just another definition.

from free alg. to constructe Weyl algebra.

Chap 2. Ideal structure. $\Rightarrow$ will show A_n is simple and contain a answer to "why we need assume k is a FIELD

degree of an operator $\Rightarrow$ tell us that it is graded.

Ideal structure $\Rightarrow$ no two-sided ideal

but for one-side it's interesting

(Unfortunately it is to advanced at this moment for us)

Positive characteristic. $\Rightarrow$ show this.

Chap 3. Rings of differential operators.

definition : what is ?, especially over $k[x]$

$A_n(k) = D(k[x])$ $\Rightarrow$ the final goal.